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Geometric Population Inversion in Rabi Oscillation

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Abstract The relations between geometric phases and population inversion in Rabi oscillation are investigated for all possible cases. The results show that the population inverse is an elliptically symmetric distribution as a function of the difference of geometric phases so as resiliently to rebut certain types of computational and experiment errors in geometric quantum computation.

Keywords Geometric phase · Population inversion · Rabi oscillation

1 Introduction

It is known that a quantum state may retain a memory of its motion in terms of a geometric phase [1–6], when it undergoes a closed evolution in parameter space. In fact, the geometric phase essentially arises as an effect of parallel transport in the Poincaré representation of the manifold [7–11].

As early as 1956, Pancharatnam [1] anticipated a quantum geometric phase. Surprisingly nobody paid much attention to the existence of the geometric phase until 1984 when Berry [2] rediscovered the phase for a quantum system under adiabatic evolution. Aharonov and Anandan [3] subsequently regarded the geometric phase factor as an inherent geometric property of the physical motion of the quantum state in the system itself and dropped the requirement for adiabatic condition. The geometric phase has been observed in spin systems through nuclear- magnetic- resonance (NMR) experiments [12] and with polarized photons

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using interferometers (PPI) [13, 14]. Recently, the geometric phase has been attracting increasing interest because of its importance for understanding and implementing geometric quantum computation [15–18] in real physical systems.

On the other hand, it is known that the current quantum computation presents a wide range of challenges to quantum information [19, 20], particularly the search for candidates for qubits who are applied to construct quantum gates. Semiconductor quantum dots are attractive because they possess energy structures and coherent optical properties similar to, and dipole moments larger than, those of atoms [21–24]. Efforts in the past few years have led to successful observations of Rabi oscillations of excitonic states [25–28], the hallmark for optical manipulation of qubits in quantum dots. Therefore, it is interesting to study the relation between the geometric phase and population inversion in Rabi oscillation. Because the definition of geometric phase for the mixed states in open system has been a controversial issue [9–11, 29–35], we start from the geometric phase of pure states.

2 Population Inversion in Rabi Oscillation

Consider the interaction of a laser field with a two-level atom. Let $|0\rangle$ and $|1\rangle$ represent the upper and lower level states of the atom, i.e., they are eigenstates of the unperturbed part of the Hamiltonian $\mathcal{H}_0 = \hbar\omega_0|0\rangle\langle 0| + \hbar\omega_0|1\rangle\langle 1|$. Thus, the state vector of a two-level atom may be expressed as

$$|\psi(\mathrm{r}, t)\rangle = c_0(t)|0\rangle e^{-i\omega_0 t} + c_1(t)|1\rangle e^{-i\omega_1 t}, \tag{1}$$

where $|c_0(t)|^2$ and $|c_1(t)|^2$ are the probability amplitudes of finding the atom in states $|0\rangle$ and $|1\rangle$, respectively. The corresponding Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi(\mathrm{r}, t)\rangle = \mathcal{H}|\psi(\mathrm{r}, t)\rangle, \tag{2}$$

with

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V}, \mathcal{V} = \mathcal{V}_{10}|1\rangle\langle 0| + \mathcal{V}_{01}|0\rangle\langle 1|, \tag{3}$$

where $\mathcal{V}$ represents the interaction of the atom with the Laser field. $\mathcal{V}_{10} = \mathcal{V}_{01}^*$ are the parts of interacting energy with

$$\mathcal{V}_{10} = \int u_1^*(\mathrm{r})\mathcal{V}u_0(\mathrm{r})dt = -\frac{1}{2}\mu E(t)e^{-i\omega_L t}. \tag{4}$$

Here $u_0(\mathrm{r})$ and $u_1(\mathrm{r})$ are two eigenstates of the Hamiltonian $\mathcal{H}$, respectively. While μ is the matrix element of the electric dipole moment and $E(t)$ is the laser field at the atom and ω_L is the frequency of laser field. Under the case of resonance $\omega_L = \omega_1 - \omega_0$, the equations of motion for the complex probability amplitudes can be obtained from the Schrödinger equation to be

$$\frac{d}{dt}\begin{pmatrix} c_1(t) \\ c_0(t) \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & \Omega(t) \\ \Omega^*(t) & 0 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_0(t) \end{pmatrix}, \tag{5}$$

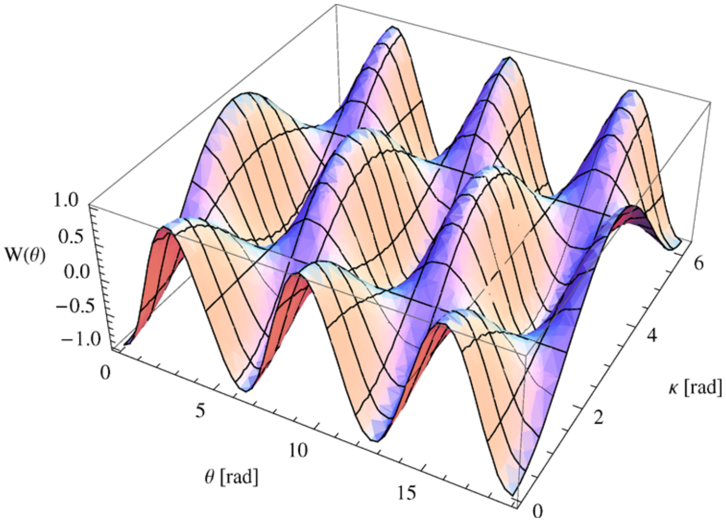

Fig. 1 Oscillations of the population inversion $W(\theta,\kappa)$ as functions of pulse area θ and initial angle κ. The results show that, by controlling the external pulse area and initial azimuthal angle of the atom, one can effectively implement population inversion in Rabi oscillation

where $\Omega(t) = \Omega^*(t) = \mu E(t)/\hbar$ is Rabi frequency. Under the initial conditions $c_0(0) = \cos(\kappa/2)$ and $c_0(0) = \sin(\kappa/2)$ with an initial azimuthal angle κ, the solution of (5) may be expressed as

$$c_0(t) = \cos\frac{\theta(t)}{2}\cos\frac{\kappa}{2} + i\sin\frac{\theta(t)}{2}\sin\frac{\kappa}{2}, \tag{6}$$

$$c_1(t) = \cos\frac{\theta(t)}{2}\sin\frac{\kappa}{2} + i\sin\frac{\theta(t)}{2}\cos\frac{\kappa}{2}, \tag{7}$$

where $\theta(t) = \int_0^t \Omega(t')dt'$ is just a pulse area regarded usually as a slowly varying function of time and position. It is noted in (6) and (7) that the normalized condition $|c_0(t)|^2 + |c_1(t)|^2 = 1$ is satisfied. Thus, the wave function (1) may be rewritten as

$$|\psi(\mathrm{r},t)\rangle = \exp\left(-i\frac{\omega_0+\omega_1}{2}t\right)\left[c_0(t)|0\rangle\exp\left(i\frac{\omega_1-\omega_0}{2}t\right) + c_1(t)|1\rangle\exp\left(-i\frac{\omega_1-\omega_0}{2}t\right)\right]. \tag{8}$$

It is worth noting that the phase factor $\exp(-i\frac{\omega_0+\omega_1}{2}t)$ is an overall phase factor, which do not produce any observable effects according to probability wave interpretation. Therefore, it may be dropped off in quantum computation. $\frac{\omega_1-\omega_0}{2}$ is frequency of Rabi oscillation. From (8), we know that the probabilities of the atom is in states $|0\rangle$ and $|1\rangle$ at time t are given by $|c_0(t)|^2$ and $|c_1(t)|^2$, respectively. Thus, the population inversion may be expressed as

$$W(t) = |c_1(t)|^2 - |c_0(t)|^2 = \sin^2\frac{\kappa}{2} - \cos^2\frac{\kappa}{2} + 2\cos\kappa\sin^2\frac{\theta(t)}{2}. \tag{9}$$

The population inversion $W(t)$ as functions of pulse area $\theta(t)$ and initial azimuthal angle κ is shown at Fig. 1. From Fig. 1, we see that $W(t)$ oscillates at the range of

$W(t) \in [0, 1]$, which depends on the pulse area $\theta(t)$ and initial azimuthal angle κ. When $W(t) = 1$, the atom is at the excited sate $|1\rangle$. Therefore, the atom is completely inversed. While $W(t) = -1$, the atom is at the state $|0\rangle$ and is not inversed. When $-1 < W(t) < 1$, the atom is at a superposition, which is a coherent state [36] called as the qubit in the quantum information. Thus, the results show that, by controlling the external pulse area $\theta(t)$ and initial azimuthal angle κ of the atom, one can effectively implement population inversion in Rabi oscillation. Therefore, it is very important to study the population inversion for searching the qubit candidates in the quantum computater.

3 Geometric Phases in Rabi Oscillation

It is known that the wave function of a quantum system retains a memory of its motion in terms of a geometric phase factor. This phase factor can be measured in principle by interfering the wave function which has undergone the above evolution with another coherent wave function that did not evolve, which enable one to discern whether or not the system has undergone an evolution [12–14] by the geometric phase. Therefore, let us study the geometric phase for the physical system in the Rabi oscillation. By setting in (8),

$$|\psi_0(t)\rangle = c_0(t)|0\rangle \exp\left(i\frac{\omega_1 - \omega_0}{2}t\right), \tag{10}$$

$$|\psi_1(t)\rangle = c_1(t)|1\rangle \exp\left(-i\frac{\omega_1 - \omega_0}{2}t\right), \tag{11}$$

the geometric phases (Pancharatnam phase) for the state $|\psi_0(t)\rangle$ may be obtained by

$$\begin{aligned}\gamma'_g &= \arg\langle\psi_0(0)|\psi_0(t)\rangle - \Im\int_0^t \langle\psi_0(t)|\frac{\partial}{\partial t}|\psi_0(t)\rangle dt \\ &= \phi_0(t) + \frac{\omega_1 - \omega_0}{2}t - \left(\frac{\theta(t)}{4}\sin\kappa + \frac{\omega_1 - \omega_0}{2}t\cos^2\frac{\kappa}{2}\right. \\ &\quad \left. - \frac{\omega_1 - \omega_0}{2}\cos\kappa\int_0^t \sin^2\frac{\theta(t)}{2}dt\right),\end{aligned} \tag{12}$$

where $\phi_0(t) = \arctan(\tan\frac{\theta}{2}\tan\frac{\kappa}{2})$.

Under the case of cyclic motion with the cyclicity $T = \frac{2\pi}{\omega_1 - \omega_0}$ and considering that $\theta(t)$ is a slowly varying function of time, the geometric phase is rewritten as

$$\gamma'_g = \phi_0(\theta) - \left(\frac{\theta}{4}\sin\kappa + 2\pi\cos^2\frac{\kappa}{2} - 2\pi\cos\kappa\sin^2\frac{\theta}{2}\right), \tag{13}$$

where the constant factor 2π is dropped off in (12), which is not important in the quantum computation. The geometric phase γ'_g is obviously dependent on both the pulse area θ and initial angle κ. The geometric phase γ'_g as functions of the pulse area θ and initial angle is shown at Fig. 2. The results show that the geometric phase γ'_g complicatedly oscillates with the increase of the pulse area θ and initial angle κ, which is helpful to control the geometric phase by adjusting the external parameter θ and initial condition κ.

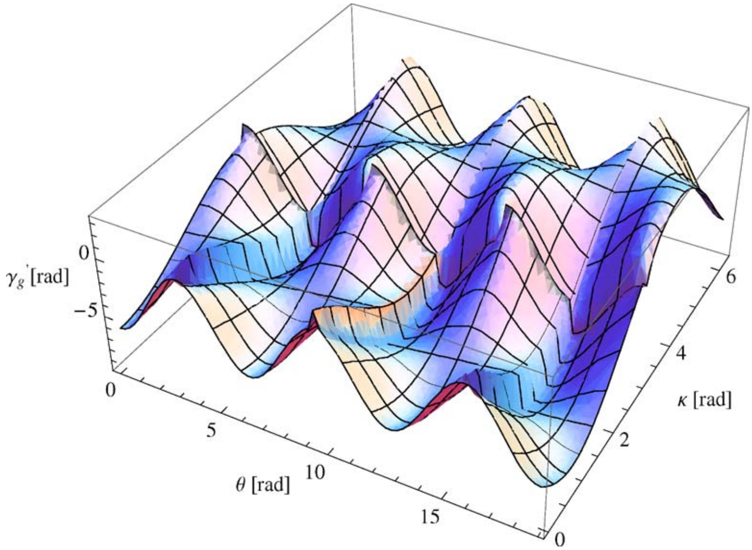

Fig. 2 Geometric phase γ_g' for the state $|\psi_0(t)\rangle$ as functions of pulse area $\theta(t)$ and initial angle κ. The results show that the geometric phase γ_g' complicatedly oscillates with the increase of the pulse area θ and initial angle κ

Similarly, another geometric phase for the state $|\psi_1(t)\rangle$ may be obtained by

$$\begin{aligned}\gamma_g'' &= \arg\langle\psi_1(0)|\psi_1(t)\rangle - \Im\int_0^t \langle\psi_1(t)|\frac{\partial}{\partial t}|\psi_1(t)\rangle dt \\ &= \phi_1(t) - \frac{\omega_1-\omega_0}{2}t \\ &\quad - \left(\frac{\theta(t)}{4}\sin\kappa - \frac{\omega_1-\omega_0}{2}t\sin^2\frac{\kappa}{2} - \frac{\omega_1-\omega_0}{2}\cos\kappa\int_0^t \sin^2\frac{\theta(t)}{2}dt\right), \end{aligned} \tag{14}$$

where $\phi_1(t) = \arctan(\tan\frac{\theta}{2}/\tan\frac{\kappa}{2})$.

Under the cyclic motion, similarly, the geometric phase for the state $|\psi_1(t)\rangle$ becomes

$$\gamma_g'' = \phi_1(\theta) - \left(\frac{\theta}{4}\sin\kappa - 2\pi\sin^2\frac{\kappa}{2} - 2\pi\cos\kappa\sin^2\frac{\theta}{2}\right), \tag{15}$$

where the pulse area θ is considered as a slowly varying function of time and the constant factor 2π is removed. The geometric phase γ_g'' for the $|\psi_1(t)\rangle$ as functions of pulse area θ and initial angle κ is shown at Fig. 3.

Comparing Fig. 2 with Fig. 3, we find that a more complex oscillations is taken place for the geometric phase γ_g' than for the geometric phase γ_g''.

For the total geometric phase γ_g for the physical system with the state $|\psi(t)\rangle = |\psi_0(t)\rangle + |\psi_1(t)\rangle$, which corresponds to remove overall phase factor in (8), we find

$$\begin{aligned}\gamma_g &= \arg\langle\psi(0)|\psi(t)\rangle - \Im\int_0^t \langle\psi(t)|\frac{\partial}{\partial t}|\psi_0(t)\rangle dt \\ &= \phi_0(t) + \phi_1(t) - \left(\frac{\theta(t)}{2}\sin\kappa + \frac{\omega_1-\omega_0}{2}t\cos\kappa - (\omega_1-\omega_0)\cos\kappa\int_0^t \sin^2\frac{\theta(t)}{2}dt\right). \end{aligned} \tag{16}$$

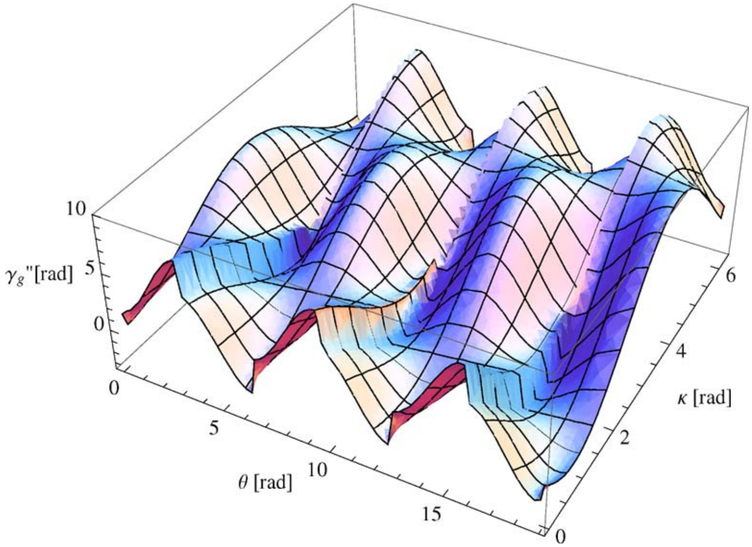

Fig. 3 Same as Fig. 2 for the geometric phase γ_g'' in the state $|\psi_1(t)\rangle$ as functions of pulse area θ and initial angle κ

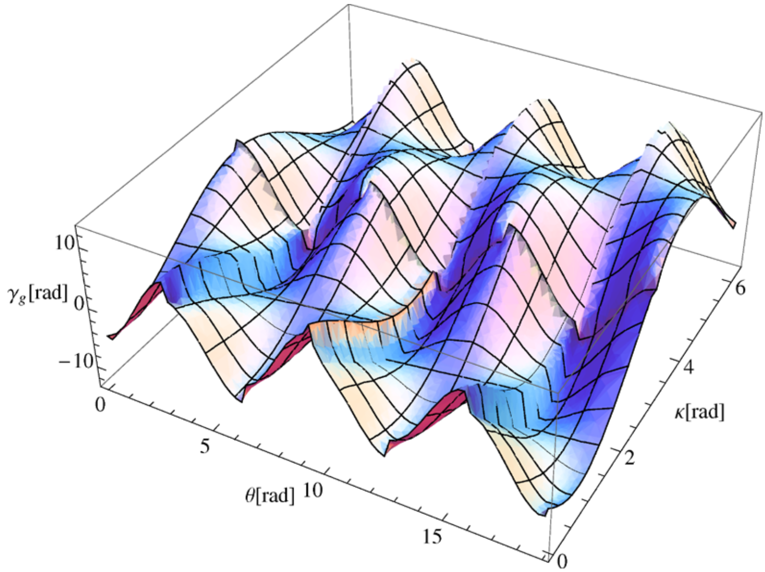

Fig. 4 Same as Figs. 2 and 3 for the total geometric phase γ_g with the state vector $|\psi(t)\rangle = |\psi_0(t)\rangle + |\psi_1(t)\rangle$, where $\gamma_g = \gamma_g' + \gamma_g''$

Under the cyclic motion, the total geometric phase is expressed as

$$\gamma_g = \phi_0(\theta) + \phi_1(\theta) - \left(\frac{\theta}{2}\sin\kappa + 2\pi\cos\kappa - 4\pi\cos\kappa\sin^2\frac{\theta}{2}\right), \tag{17}$$

where the constant factor 2π is removed. γ_g is similar to Aharonov and Anandan phase. The total geometric phase γ_g is shown at Fig. 4 as functions of the pulse area θ and initial angle κ. It is noted that, obviously, it includes the characteristic both the γ_g' and γ_g'' because of $\gamma_g = \gamma_g' + \gamma_g''$.

4 Population Inversion Via Geometric Phase

Methods for the physical implementation of quantum computation via the geometric phase have been proposed [37–42], in terms of the all-geometric approaches called holonomic quantum computations, with the feature that one can achieve the entangling universal quantum gates based entirely on purely geometric operations (holonomies) [16–18]. The holonomies are built up when a quantum system is driven to a cyclic evolution through adiabatic or nonadiabatic change in the control parameters in the Hamiltonian. Therefore, The holonomic quantum computation is a scheme intrinsically fault-tolerant and therefore resilient to certain types of computational errors. Thus, it is important to study the relation between the geometric phase and population inversion.

From (9) and (17), we see that both the population inversion $W(\theta)$ and the total geometric phase $\gamma_g(\theta)$ for our physical system in Rabi oscillation all are a function of the pulse area θ. Therefore, it is interesting to show their relations by the parameterized figures, which are shown at Figs. 5 and 6 for the different initial angles. From Figs. 5 and 6, we see that $W(\theta)$ is linear functions of total geometric phase $\gamma_g(\theta)$. Unfortunately, they are not singlet-value

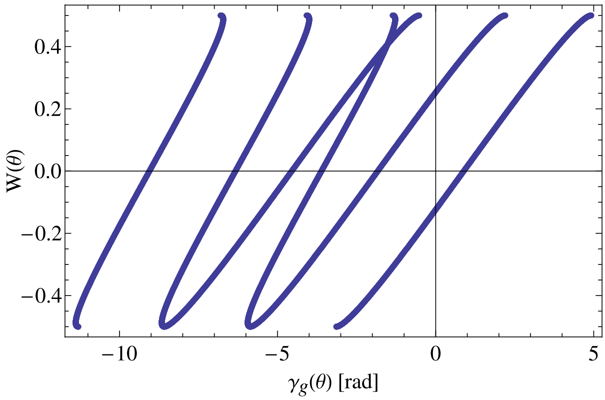

Fig. 5 The parameter plot for the population inversion $W(\theta)$ as a function of the total geometric phase $\gamma_g(\theta)$ with the initial angle $\kappa = \pi/3$. The results show that they almost are linear relations with different two-group parallel lines, where the connecting dots of different two-group parallel lines are minimum values. However, they are not single-value functions

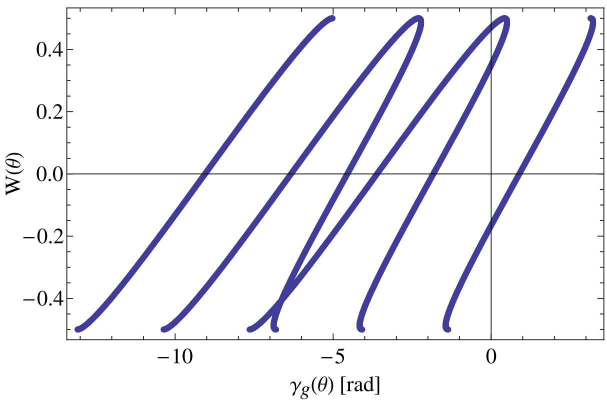

Fig. 6 Same as Fig. 5 with the initial angle $\kappa = 2\pi/3$, where the connecting dots of different two-group parallel lines are maximum values

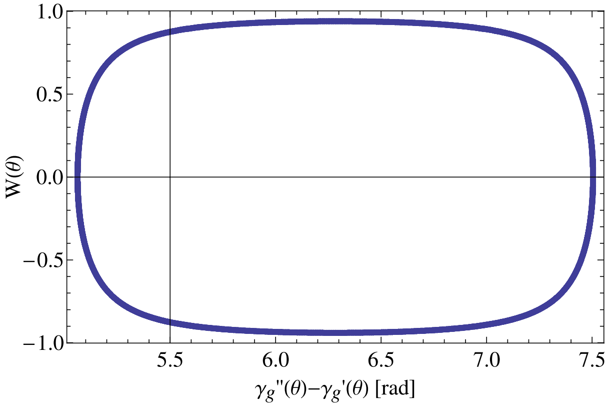

Fig. 7 The parameter plot for the population inversion $W(\theta)$ as a function of the geometric phase difference $\gamma_g''(\theta) - \gamma_g'(\theta)$ with the initial angle $\kappa = \pi/9$, which is an elliptical function

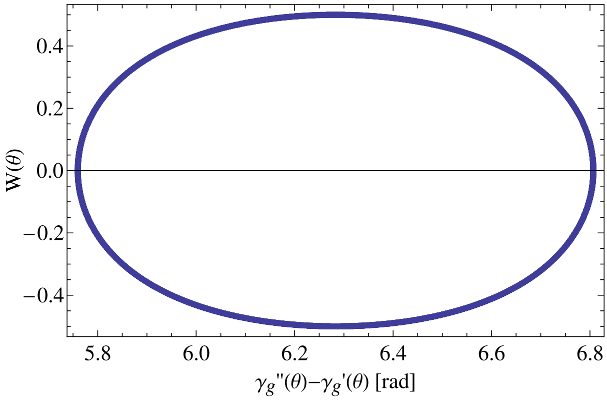

Fig. 8 Same as Fig. 7 with the initial angle $\kappa = \pi/3$

functions. It is interesting to noted that there exist different parallel lines of two groups and there are connecting dots for the different group parallel lines. The dots emerge as minimum values (See Fig. 5) or maximum values (See Fig. 6) for the different initial angles, which are obvious to be possibly used as a physical quantity described the population inversion $W(\theta)$ in the total geometric phases $\gamma_g(\theta)$.

It appears that it is slightly complicated for the relation between the population inverse $W(\theta)$ and total geometric phase $\gamma_g(\theta) = \gamma_g'(\theta) + \gamma_g''(\theta)$. In the following, therefore, we take the population inversion in Rabi oscillation as a function of the difference $\gamma_g''(\theta) - \gamma_g'(\theta)$ of geometric phases, which is shown at Figs. 7 and 8 for different initial angles. it is surprised to find that the population inversion is an elliptically symmetric distribution as a function of the difference $\gamma_g''(\theta) - \gamma_g'(\theta)$ of geometric phases. The elliptical shape and parameters are, certainly, dependent on the initial angle (see Figs. 7 and 8). The elliptically symmetric distribution may be important for the geometric quantum computation because it map the physical quantity onto the geometry.

5 Conclusion

Coherent manipulation of quantum states is a critical step toward many novel technological applications ranging from operating of qubits in quantum logic gates to controlling the reaction pathways of molecules. The Rabi oscillation plays central roles in coherent control. Geometric quantum computation uses geometric property to do quantum computation.

In our work, a scheme is proposed to manipulate the Rabi oscillation by the difference $\gamma_g''(\theta) - \gamma_g'(\theta)$ of geometric phases, which is potentially intrinsically fault tolerant and therefore resilient to certain types of experimental and computational errors.

In conclusion, we investigate the relations between geometric phases and population inversion in Rabi oscillation. The results show that the population inverse $W(\theta)$ is an elliptically symmetric distribution as a function of the difference $\gamma_g''(\theta) - \gamma_g'(\theta)$ of geometric phases, which is very helpful to operating the Rabi oscillation by the geometric phases.

References

1. Pancharatnam, S.: Proc. Ind. Acad. Sci. A **44**, 1225 (1956)
2. Berry, M.V.: Proc. R. Soc. Lond., Ser. A **392**, 45 (1984)
3. Aharonov, Y., Anandan, J.: Phys. Rev. Lett. **58**, 1593 (1987)
4. Simon, B.: Phys. Rev. Lett. **51**, 2167 (1983)
5. Wang, Z.S., et al.: Phys. Scr. **75**, 494 (2007)
6. Wang, Z.S., et al.: Eur. Phys. J. D **33**, 285 (2005)
7. Byrd, M.: J. Math. Phys. **39**, 6125 (1998)
8. Strahov, E.: J. Math. Phys. **42**, 2008 (2001)
9. Wang, Z.S., et al.: Europhys. Lett. **74**, 958 (2006)
10. Wang, Z.S., et al.: Phys. Rev. A **75**, 024102 (2007)
11. Wang, Z.S.: Int. J. Theor. Phys. **48**, 2353 (2009)
12. Suter, D., Mueller, K.T., Pine, A.: Phys. Rev. Lett. **60**, 1218 (1988)
13. Bhandari, R., Samuel, J.: Phys. Rev. Lett. **60**, 1211 (1988)
14. Chiao, R.Y., et al.: Phys. Rev. Lett. **60**, 1214 (1988)
15. Zanardi, P., Rasetti, M.: Phys. Lett. A **264**, 94 (1999)
16. Wang, Z.S., et al.: Phys. Rev. A **76**, 044303 (2007)
17. Wang, Z.S.: Phys. Rev. A **79**, 024304 (2009)
18. Wang, Z.S., Liu, G.Q., Ji, Y.H.: Phys. Rev. A **79**, 054301 (2009)
19. Bennett, C.H., DiVincenzo, D.P.: Nature (London) **404**, 247 (2000)
20. Barenco, A., Deutsch, D., Ekert, A., Jozsa, R.: Phys. Rev. Lett. **74**, 4083 (1995)
21. Gammon, D., et al.: Science **273**, 87 (1996)
22. Bayer, M., et al.: Nature (London) **405**, 923 (2000)
23. Chen, G.: Science **289**, 1906 (2000)
24. Bonadeo, N.H., et al.: Science **282**, 1473 (1998)
25. Zrenner, A., et al.: Nature (London) **418**, 612 (2002)
26. Borri, P.: Phys. Rev. B **66**, 081306(R) (2002)
27. Stievater, T.H.: Phys. Rev. Lett. **87**, 133603 (2001)
28. Wang, Q.Q., et al.: Phys. Rev. B **72**, 035306 (2005)
29. Sjöqvist, E., et al.: Phys. Rev. Lett. **85**, 2845 (2000)
30. Carollo, A., Fuentes-Guridi, I., Franca Santos, M., Vedral, V.: Phys. Rev. Lett. **90**, 160402 (2003)
31. Fonseca Romero, K.M., et al.: Physica A **307**, 142 (2002)
32. Nazir, A., et al.: Phys. Rev. A **65**, 042303 (2003)
33. Whitney, R.S., Gefen, Y.: Phys. Rev. Lett. **90**, 190402 (2003)
34. De Chiara, G., Palma, M.: Phys. Rev. Lett. **91**, 090404 (2003)
35. Whitney, R.S., et al.: Phys. Rev. Lett. **94**, 070407 (2005)
36. Wang, Z.S., Wu, R.S.: Int. J. Theor. Phys. **48**, 1859 (2009)
37. Wang, Z.S., et al.: Phys. Lett. A **359**, 608 (2006)
38. Wang, Z.S., et al.: Phys. Lett. A **372**, 775 (2008)
39. Leibfried, D., et al.: Nature **422**, 422 (2003)
40. Zhu, S.L., Wang, Z.D.: Phys. Rev. Lett. **91**, 187902 (2003)
41. Chen, C.Y., Feng, M., Zhang, X.L., Gao, K.L.: Phys. Rev. A **73**, 032344 (2006)
42. Fuentes-Guridi, I., Bose, S., Vedral, V.: Phys. Rev. Lett. **85**, 5018 (2000)